

WEAK SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATION INVOLVING DIRAC MASS

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Abstract. In this paper, we study the following elliptic problem with Dirac mass

$$\begin{cases} -\Delta u = Vu^p + k\delta_0 & \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow +\infty} u(x) = 0, \end{cases} \quad (1)$$

where $N > 2$, $p > 0$, $k > 0$, δ_0 is Dirac mass at the origin, the function V is a locally Lipschitz continuous in $\mathbb{R}^N \setminus \{0\}$ satisfying

$$V(x) \leq \frac{c_1}{|x|^{a_0}(1 + |x|^{a_\infty - a_0})}$$

with $a_0 < N$, $a_\infty > a_0$ and $c_1 > 0$. We obtain two positive solutions of (1) with additional conditions for parameters on a_∞, a_0, p and k . The first solution is a minimal positive solution and the second solution is constructed by Mountain Pass theorem.

1. INTRODUCTION

The main objective of this paper is to study the existence of multiple weak solutions to the following nonlinear elliptic problem with Dirac mass

$$(P_k) \quad \begin{cases} -\Delta u = Vu^p + k\delta_0 & \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow +\infty} u(x) = 0 \end{cases}$$

where $N > 2$, $p > 0$, $k > 0$, δ_0 is Dirac mass at the origin, the weight function V is locally Lipschitz continuous in $\mathbb{R}^N \setminus \{0\}$. Problem (P_k) concerns with source term in contrast with problems with absorption terms. The semi-linear elliptic equations with absorption terms

$$\begin{cases} -\Delta u + g(u) = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where ν is a bounded Radon measure, Ω is a bounded C^2 domain in \mathbb{R}^N and $g : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and $g(0) \geq 0$, has been extensively studied for the last several decades. A fundamental contribution to the problem is due to Brezis [8], Benilan and Brezis [5], where they showed the existence and uniqueness of weak solution for problem (1.1) if the function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s))s^{-1-\frac{N}{N-2}} ds < +\infty.$$

The method is to approximate the measure ν by a sequence of regular functions, and find classical solutions which converges to a weak solution of (1.1). Meanwhile, it is necessary to establish uniform bounds for the sequence of classical solutions. The uniqueness is then

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derived by Kato's inequality. Such a method has been extended to solve the equations with boundary measure data in [13, 16, 17, 18, 19] and other subjects in [3, 4, 6, 7].

In the source term case, one adapts different approaches since it is hard to find uniform bound if one uses the approaching process in [5, 8]. Moreover, it seems that the uniqueness is no longer valid in general. Actually, for the problem

$$\begin{cases} -\Delta u = u^q + \lambda \delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $q \in (1, \frac{N}{N-2})$, $\lambda > 0$ and Ω is a bounded domain containing the origin, it was shown in [15] that there exists $\lambda^* > 0$ such that (1.2) has two solutions for $\lambda \in (0, \lambda^*)$. For general Radon measures μ , one weak solution was found in [4] for (1.2) with δ_0 replaced by μ . If $1 < q < \frac{N}{N-2}$, the solutions of (1.2) are isolated singular solutions of

$$-\Delta u = u^q \quad \text{in } \Omega \setminus \{0\}, \quad (1.3)$$

such solutions asymptotically behave at the origin like $|x|^{2-N}$. The nonnegative solutions to (1.3) with isolated singularities have been classified in [1] for $q = \frac{N}{N-2}$, in [12] for $\frac{N}{N-2} < q < \frac{N+2}{N-2}$ and in [9] for $q = \frac{N+2}{N-2}$. Using this classification of singular solutions, one may construct solutions of the equation like (1.3) with many singular points, see for instance [20, 23].

In the whole space, it was proved in [22] that the problem

$$\begin{cases} -\Delta u + u = u^q + \kappa \sum_{i=1}^m \delta_{x_i} & \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \lim_{|x| \rightarrow +\infty} u(x) = 0 \end{cases} \quad (1.4)$$

possesses at least two weak solutions for $\kappa > 0$ small and $q \in (1, \frac{N}{N-2})$. A feature of the operator $-\Delta + id$ is that its fundamental solution decays exponentially at infinity, thus the fundamental solution belongs to $L^1(\mathbb{R}^N)$. This fact plays an essential role in finding solutions of (1.4). While in our problem (P_k) , the fundamental solution of $-\Delta$ does not belong to $L^1(\mathbb{R}^N)$. It brings difficulties in the process of finding solutions of (P_k) .

In this paper, we will find two weak solutions for problem (P_k) . By a *weak solution* of (P_k) we mean a nonnegative function $u \in L^1_{loc}(\mathbb{R}^N)$ such that $Vu^p \in L^1(\mathbb{R}^N)$,

$$\lim_{r \rightarrow +\infty} \sup_{x \in \mathbb{R}^N \setminus B_r(0)} u(x) = 0$$

and u satisfies

$$\int_{\mathbb{R}^N} u(-\Delta)\xi dx = \int_{\mathbb{R}^N} Vu^p \xi dx + k\xi(0), \quad \forall \xi \in C_c^{1,1}(\mathbb{R}^N).$$

We suppose throughout this paper that there exist $a_0 < N$, $a_\infty > a_0$, $c_1 > 0$ such that the function $V(x)$ satisfies

$$V(x) \leq V_0(x) := \frac{c_1}{|x|^{a_0}(1 + |x|^{a_\infty - a_0})}. \quad (1.5)$$

Condition (1.5) implies that the limiting behavior of V at the origin is controlled by $|x|^{-a_0}$ and that of $V(x)$ at infinity controlled by $|x|^{-a_\infty}$ respectively.

The first result is on the minimal solution of (P_k) .

Theorem 1.1. *Suppose condition (1.5) holds and $p > 0$ satisfies*

$$p \in \left(\frac{N - a_\infty}{N - 2}, \frac{N - a_0}{N - 2} \right). \quad (1.6)$$

Then,

(i) there exists $k^* = k^*(p, V) \in (0, +\infty]$ such that for $k \in (0, k^*)$, there exists a minimal positive solution $u_{k,V}$ of (P_k) and for $p > 1$ and $k > k^*$, there is no solution for (P_k) . Moreover, we have $k^* < \infty$ if $p > 1$; $k^* = +\infty$ if $0 < p < 1$ or $p = 1$ and $c_1 > 0$ small.

(ii) for p fixed, the mapping $V \mapsto k^*$ is decreasing and the mapping $V \mapsto u_{k,V}$ is increasing.

(iii) if V is radially symmetric, the minimal solution $u_{k,V}$ is also radially symmetric.

In the sequel, we denote $u_{k,V}$ the minimal solution obtained in Theorem 1.1 corresponding to k and V .

We remark that the minimal solution of (P_k) is derived by iterating an increasing sequence $\{v_n\}_n$ defined by

$$v_0 = k\mathbb{G}[\delta_0], \quad v_n = \mathbb{G}[Vv_{n-1}^p] + k\mathbb{G}[\delta_0],$$

where $\mathbb{G}[\cdot]$ is the Green operator defined as

$$\mathbb{G}[f](x) = \int_{\mathbb{R}^N} G(x, y)f(y)dy$$

and G is the Green kernel of $-\Delta$ in $\mathbb{R}^N \times \mathbb{R}^N$. It is known that $\mathbb{G}[\delta_0]$ is the fundamental solution of $-\Delta$. To insure the convergence of the sequence $\{v_n\}_n$, we need to construct a suitable barrier function by using the estimate

$$\mathbb{G}[V\mathbb{G}^p[\delta_0]] \leq c_2\mathbb{G}[\delta_0] \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad (1.7)$$

where $c_2 > 0$. From the estimate (1.7), the optimal range of k for constructing super solution of (P_k) is $(0, k_p]$, where

$$k_p = (c_2 p)^{-\frac{1}{p-1}} \frac{p-1}{p}. \quad (1.8)$$

Here we observe that $k^* \geq k_p$.

Once the minimal solution is found, we explore further properties of the solution. Precisely, we show such a solution is regular except for the origin, and decays at infinity. These properties allow us to establish the stability of the minimal solution, whereas this stability plays the role in finding the second solution.

Denote by $\mathcal{D}^{1,2}(\mathbb{R}^N)$ the Sobolev space which is the closure of $C_c^\infty(\mathbb{R}^N)$ under the norm

$$\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^{\frac{1}{2}}.$$

We say a solution u of (P_k) is *stable* (resp. *semi-stable*) if

$$\int_{\mathbb{R}^N} |\nabla \xi|^2 dx > p \int_{\mathbb{R}^N} V u^{p-1} \xi^2 dx, \quad (\text{resp. } \geq) \quad \forall \xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}.$$

Let k_p be given in (1.8). The properties of the minimal solution are collected as follows.

Theorem 1.2. *Suppose that the function V satisfies (1.5) with $a_\infty > a_0$ and $a_0 \in \mathbb{R}$, and p satisfies (1.6).*

(i) *If $a_0 < 2$, $p > 1$ and $k \in (0, k_p)$, then $u_{k,V}$ is a classical solution of the equation*

$$-\Delta u = V u^p \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad \lim_{|x| \rightarrow +\infty} u(x) = 0 \quad (1.9)$$

and satisfies

$$\sup_{x \in \mathbb{R}^N \setminus \{0\}} u(x) |x|^{N-2} < +\infty. \quad (1.10)$$

Moreover, $u_{k,V}$ is stable and there exists $c_3 > 0$ independent of k such that

$$\int_{\mathbb{R}^N} |\nabla \xi|^2 dx - p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx \geq c_3 [(k^*)^{\frac{p-1}{p}} - k^{\frac{p-1}{p}}] \int_{\mathbb{R}^N} |\nabla \xi|^2 dx, \quad \forall \xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}. \quad (1.11)$$

(ii) If

$$p \in \left(0, \frac{N}{N-2}\right) \quad (1.12)$$

and $k \in (0, k^*)$, the minimal solution $u_{k,V}$ is stable and satisfies (1.11). Moreover, any positive weak solution u of (P_k) is a classical solution of problem (1.9) satisfying (1.10).

We note that in (i) and (ii) of Theorem 1.2, the parameter k is bounded by k_p and k^* respectively. It is not clear if $k_p < k^*$. We also remark that (1.10) implies that the singularity and the decays of u at the origin and infinity respectively are the same as the fundamental solution.

The second solution of (P_k) will be constructed by Mountain Pass Theorem. Indeed, we will look for critical points of the functional

$$E(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} V F(u_{k,V}, v_+) dx \quad (1.13)$$

in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, where $t_+ = \max\{0, t\}$,

$$F(s, t) = \frac{1}{p+1} [(s+t_+)^{p+1} - s^{p+1} - (p+1)s^p t_+].$$

To assure that the functional E is well defined, we establish the embedding

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, V_0 u_{k,V}^{p-1} dx) \quad (1.14)$$

and

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N, V_0 dx), \quad (1.15)$$

both of them are compact if

$$p+1 \in (2^*(a_\infty), 2^*(a_0)) \cap [1, 2^*), \quad (1.16)$$

where $2^*(t) = \frac{2N-2t}{N-2}$ with $t \in \mathbb{R}$ and $2^* = 2^*(0)$. Therefore, we may verify that the functional E satisfies the $(PS)_c$ condition. Furthermore, we may build the mountain pass structure by the stability of the minimal solution.

Taking into account the range of p for the existence of the minimal solution, we suppose

$$p \in \left(\frac{N-a_\infty}{N-2}, \frac{N-a_0}{N-2}\right) \cap (\max\{2^*(a_\infty) - 1, 0\}, \min\{2^*(a_0) - 1, 2^* - 1\}). \quad (1.17)$$

The intersection of intervals in (1.17) is not empty if we assume further that

$$a_0 < 2, \quad a_\infty > \max\{0, 1 + \frac{a_0}{2}\}. \quad (1.18)$$

Our result of the existence of the second solution is stated as follows.

Theorem 1.3. *Suppose that the function V satisfies (1.5) with a_0 and a_∞ given in (1.18), $p > 1$ satisfies (1.17) and k_p is given by (1.8). Then, for $k \in (0, k_p)$, problem (P_k) admits a weak solution $u > u_{k,V}$. Moreover, both u and $u_{k,V}$ are classical solutions of (1.9).*

Although we are not able to show $k_p < k^*$, we may prove that if p satisfies (1.12), problem (P_k) admits a solution u such that $u > u_{k,V}$ for all $k \in (0, k^*)$.

If V is radially symmetric, the range of p can be improved to

$$p \in \left(\frac{N-a_\infty}{N-2}, \frac{N-a_0}{N-2}\right) \cap (\max\{2^*(a_\infty) - 1, 0\}, 2^*(a_0) - 1). \quad (1.19)$$

Finally, we have the following result for V being radial.

Theorem 1.4. *Suppose that the function V is radially symmetric satisfying (1.5) with a_0 and a_∞ given in (1.18), $p > 1$ satisfies (1.19) and k_p is given by (1.8). Then, for $k \in (0, k_p)$, problem (P_k) admits a radially symmetric solution $u > u_{k,V}$, and both u and $u_{k,V}$ are classical solutions of (1.9).*

The paper is organized as follows. In Section 2, we show the existence of the minimal solution of (P_k) . Section 3 is devoted the regularity and stability of the minimal solution. Finally, in Section 4 we find the second solution of (P_k) by the Mountain Pass theorem.

2. MINIMAL SOLUTION

In this section, we show the existence of the minimal solution for (P_k) . To this end, we construct a monotone sequence of approximating solutions by the iterating technique and bound it by a super-solution. The suitable super-solution will be constructed based on the following result.

Lemma 2.1. *Assume that the function V satisfies (1.5) with $a_0 < N$, $a_\infty > 0$, and that $p > 0$ satisfies (1.6). There holds*

$$\mathbb{G}[V\mathbb{G}^p[\delta_0]] \leq c_2 \mathbb{G}[\delta_0] \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad (2.1)$$

where c_2 linearly depends on c_1 .

Proof. Note that

$$\mathbb{G}[\delta_0](x) = \frac{c_N}{|x|^{N-2}}, \quad (2.2)$$

by the assumption for p , we have

$$V(x)\mathbb{G}^p[\delta_0](x) \leq \frac{c_N^p c_1}{|x|^{(N-2)p+a_0}(1+|x|^{a_\infty-a_0})}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \quad (2.3)$$

where $c_N > 0$ is the normalized constant depending only on N . This implies

$$V\mathbb{G}^p[\delta_0] \in L^1(\mathbb{R}^N).$$

We deduce by (2.2) and (2.3) that

$$\begin{aligned} & \mathbb{G}[V\mathbb{G}^p[\delta_0]](x) \\ & \leq c_N^{p+1} c_1 \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{1}{1+|y|^{a_\infty-a_0}} \frac{1}{|y|^{(N-2)p+a_0}} dy \\ & = c_N^{p+1} c_1 |x|^{2-(N-2)p-a_0} \int_{\mathbb{R}^N} \frac{1}{|e_x - y|^{N-2}} \frac{1}{1+|x|^{a_\infty-a_0}|y|^{a_\infty-a_0}} \frac{1}{|y|^{(N-2)p+a_0}} dy \\ & := c_N^{p+1} c_1 |x|^{2-(N-2)p-a_0} \int_{\mathbb{R}^N} \Phi(x, y) dy, \end{aligned}$$

where $e_x = \frac{x}{|x|}$.

Now, we estimate $\int_{\mathbb{R}^N} \Phi(x, y) dy$. We divide it into two cases (i) $|x| \geq 1$ and (ii) $|x| \leq 1$ to discuss.

In the case (i) $|x| \geq 1$, by (1.6), $(N-2)p + a_0 < N$, we have

$$\begin{aligned}
& \int_{B_{\frac{1}{2}}(0)} \Phi(x, y) dy \\
& \leq c_4 \int_{B_{\frac{1}{2}}(0)} \frac{1}{1 + |x|^{a_\infty - a_0} |y|^{a_\infty - a_0}} \frac{1}{|y|^{(N-2)p + a_0}} dy \\
& = c_4 |x|^{(N-2)p + a_0 - N} \int_{B_{\frac{|x|}{2}}(0)} \frac{1}{1 + |z|^{a_\infty - a_0}} \frac{1}{|z|^{(N-2)p + a_0}} dz \\
& \leq c_4 |x|^{(N-2)p + a_0 - N} \left(\int_{B_{\frac{1}{2}}(0)} \frac{dz}{|z|^{(N-2)p + a_0}} + \int_{B_{\frac{|x|}{2}}(0) \setminus B_{\frac{1}{2}}(0)} \frac{dz}{|z|^{(N-2)p + a_0}} \right) \\
& \leq c_5 \left(|x|^{(N-2)p + a_0 - N} + |x|^{a_0 - a_\infty} \right),
\end{aligned}$$

where $c_4, c_5 > 0$.

If $y \in B_{\frac{1}{2}}(e_x)$, we have

$$\frac{1}{1 + |x|^{a_\infty - a_0} |y|^{a_\infty - a_0}} \frac{1}{|y|^{(N-2)p + a_0}} \leq c_6 |x|^{a_0 - a_\infty},$$

and then

$$\int_{B_{\frac{1}{2}}(e_x)} \Phi(x, y) dy \leq c_7 |x|^{a_0 - a_\infty} \int_{B_{\frac{1}{2}}(e_x)} \frac{1}{|e_x - y|^{N-2}} dy \leq c_8 |x|^{a_0 - a_\infty},$$

where $c_6, c_7, c_8 > 0$.

If $y \in \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cap B_{\frac{1}{2}}(e_x))$, we have $(N-2)(p+1) + a_\infty > N$ by (1.6). Therefore,

$$\int_{\mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cap B_{\frac{1}{2}}(e_x))} \Phi(x, y) dy \leq c_9 |x|^{a_0 - a_\infty} \int_{\mathbb{R}^N \setminus B_1(0)} \frac{dy}{|y|^{(N-2)(p+1) + a_\infty}} \leq c_{10} |x|^{a_0 - a_\infty},$$

where $c_9, c_{10} > 0$.

By the assumption on p , $(N-2)p + a_\infty \geq N$. We conclude that there exists $c_{11} > 0$ such that for $|x| \geq 1$,

$$\mathbb{G}[V\mathbb{G}^p[\delta_0]](x) \leq c_{11} \max\{|x|^{2-N}, |x|^{2-(N-2)p-a_\infty}\} \leq c_{11} |x|^{2-N}. \quad (2.4)$$

Next, we treat the case (ii) $|x| \leq 1$.

Apparently, there exist $c_{12}, c_{13} > 0$ such that

$$\int_{B_{\frac{1}{2}}(0)} \Phi(x, y) dy \leq c_{12} \int_{B_{\frac{1}{2}}(0)} \frac{1}{|y|^{(N-2)p + a_0}} dy \leq c_{13}.$$

By (1.6), there are $c_{14}, c_{15} > 0$ such that

$$\int_{B_{\frac{1}{2}}(e_x)} \Phi(x, y) dy \leq c_{14} \int_{B_{\frac{1}{2}}(e_x)} \frac{1}{|e_x - y|^{N-2}} dy \leq c_{15}.$$

For $y \in \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cap B_{\frac{1}{2}}(e_x))$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cap B_{\frac{1}{2}}(e_x))} \Phi(x, y) dy \\ & \leq c_{16} \int_{\mathbb{R}^N \setminus B_1(0)} \frac{1}{1 + |x|^{a_\infty - a_0} |y|^{a_\infty - a_0}} \frac{1}{|y|^{(N-2)p + a_0 + N - 2}} dy \\ & \leq c_{16} |x|^{(N-2)p + a_0 - 2} \int_{\mathbb{R}^N \setminus B_1(0)} \frac{dz}{|z|^{(N-2)p + a_0 + N - 2} (1 + |z|^{a_\infty - a_0})} \\ & \leq c_{17} |x|^{(N-2)p + a_0 - 2}, \end{aligned}$$

where $c_{16}, c_{17} > 0$.

Consequently, there exists $c_{18} > 0$ such that for $|x| \leq 1$,

$$\mathbb{G}[V\mathbb{G}^p[\delta_0]](x) \leq c_{18} |x|^{2-N}. \quad (2.5)$$

Therefore, the assertion follows by (2.4) and (2.5). \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, we prove (i). We define the iterating sequence:

$$v_0 := k\mathbb{G}[\delta_0] > 0,$$

and

$$v_n = \mathbb{G}[Vv_{n-1}^p] + k\mathbb{G}[\delta_0]. \quad (2.6)$$

Observing that

$$v_1 = \mathbb{G}[Vv_0^p] + k\mathbb{G}[\delta_0] > v_0,$$

and assuming that

$$v_{n-1}(x) \geq v_{n-2}(x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

we deduce that

$$v_n = \mathbb{G}[Vv_{n-1}^p] + k\mathbb{G}[\delta_0] \geq \mathbb{G}[Vv_{n-2}^p] + k\mathbb{G}[\delta_0] = v_{n-1}. \quad (2.7)$$

Thus, the sequence $\{v_n\}_n$ is increasing with respect to n . Moreover, we have that

$$\int_{\mathbb{R}^N} v_n(-\Delta)\xi dx = \int_{\mathbb{R}^N} Vv_{n-1}^p \xi dx + k\xi(0), \quad \forall \xi \in C_c^{1,1}(\mathbb{R}^N). \quad (2.8)$$

Now, we build an upper bound for $\{v_n\}_n$. For $t > 0$, denote

$$w_t = tk^p \mathbb{G}[V\mathbb{G}[\delta_0]^p] + k\mathbb{G}[\delta_0] \leq (c_2 tk^p + k)\mathbb{G}[\delta_0],$$

where $c_2 > 0$ is from Lemma 2.1, then

$$\mathbb{G}[Vw_t^p] + k\mathbb{G}[\delta_0] \leq (c_2 tk^p + k)^p \mathbb{G}[V\mathbb{G}[\delta_0]^p] + k\mathbb{G}[\delta_0] \leq w_t$$

if

$$(c_2 tk^{p-1} + 1)^p \leq t. \quad (2.9)$$

Now, we choose t such that (2.9) holds.

If $p > 1$, since the function $f(t) = (\frac{1}{p}(\frac{p-1}{p})^{p-1}t + 1)^p$ intersects the line $g(t) = t$ at the unique point t_p , we may choose that

$$c_2 k^{p-1} \leq \frac{1}{p} \left(\frac{p-1}{p} \right)^{p-1} \quad \text{and} \quad t_p = \left(\frac{p}{p-1} \right)^p. \quad (2.10)$$

If $p = 1$, we choose $c_1 > 0$ small such that $c_2 < 1$ and

$$t_p = \frac{1}{1 - c_2}.$$

Finally, if $p < 1$, for $t > 1$, we have

$$(c_2 t k^{p-1} + 1)^p \leq (c_2 k^{p-1} + 1)^p t^p,$$

so we may choose

$$t_p = (c_2 k^{p-1} + 1)^{\frac{p}{1-p}}.$$

Hence, for t_p we have chosen, by the definition of w_{t_p} , we have $w_{t_p} > v_0$ and

$$v_1 = \mathbb{G}[V v_0^p] + k \mathbb{G}[\delta_0] < \mathbb{G}[V w_{t_p}^p] + k \mathbb{G}[\delta_0] = w_{t_p}.$$

Inductively, we obtain

$$v_n \leq w_{t_p} \tag{2.11}$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\{v_n\}$ converges. Let $u_{k,V} := \lim_{n \rightarrow \infty} v_n$. By (2.8), $u_{k,V}$ is a weak solution of (P_k) .

We claim that $u_{k,V}$ is the minimal solution of (P_k) , that is, for any positive solution u of (P_k) , we always have $u_{k,V} \leq u$. Indeed, there holds

$$u = \mathbb{G}[V u^p] + k \mathbb{G}[\delta_0] \geq v_0,$$

and then

$$u = \mathbb{G}[V u^p] + k \mathbb{G}[\delta_0] \geq \mathbb{G}[V v_0^p] + k \mathbb{G}[\delta_0] = v_1.$$

We may show inductively that

$$u \geq v_n$$

for all $n \in \mathbb{N}$. The claim follows.

Similarly, if problem (P_k) has a nonnegative solution u for $k_1 > 0$, then (P_k) admits a minimal solution $u_{k,V}$ for all $k \in (0, k_1]$. As a result, the mapping $k \mapsto u_{k,V}$ is increasing. So we may define

$$k^* = \sup\{k > 0 : (P_k) \text{ has a minimal solution for } k\},$$

which is the largest k such that problem (P_k) has minimal positive solution, and $k^* > 0$.

We remark that in the cases $0 < p < 1$ and $p = 1$, $c_1 > 0$ small, we may always find a super-solution w_{t_p} . Hence, there exists a minimal solution for all $k > 0$, that is, $k^* = \infty$.

Now, we prove that $k^* < +\infty$ if $p > 1$. Suppose on the contrary that, problem (P_k) admits a minimal solution $u_{k,V}$ for $k > 0$ large. We observe that

$$u_{k,V} \geq k \mathbb{G}[\delta_0].$$

Let $x_0 \in \text{supp} V$ be a point such that $x_0 \neq 0$, $V(x_0) > 0$ and for some $r > 0$ such that

$$V(x) \geq \frac{V(x_0)}{2}, \quad \forall x \in B_r(x_0).$$

Denote by η_0 a C^2 function such that

$$\eta_0(x) = 1, \quad x \in B_1(0) \quad \text{and} \quad \eta_0(x) = 0, \quad \forall x \in \mathbb{R}^N \setminus B_2(0).$$

Let $\eta_0^R(x) = \eta_0(\frac{x-x_0}{R})$ and

$$\xi_R(x) = \mathbb{G}[\chi_{B_r(x_0)}] \eta_0^R(x) \in C_c^{1,1}(\mathbb{R}^N)$$

for $R > r$, where χ_Ω is the characterization function of Ω . Thus,

$$\lim_{R \rightarrow +\infty} \xi_R = \mathbb{G}[\chi_{B_r(x_0)}].$$

Taking ξ_R as a test function with $R > 4r$, we obtain

$$\int_{B_r(x_0)} u_{k,V} dx + \int_{B_{2R}(x_0) \setminus B_R(x_0)} u_{k,V}(-\Delta)\xi_R dx = \int_{\mathbb{R}^N} V u_{k,V}^p \xi_R dx + k\xi_R(0). \quad (2.12)$$

For $x \in B_{2R}(x_0) \setminus B_R(x_0)$, we have

$$|(-\Delta)\xi_R(x)| \leq |\nabla \mathbb{G}[\chi_{B_r(x_0)}] \cdot \nabla \eta_0^R(x)| + |\mathbb{G}[\chi_{B_r(x_0)}](-\Delta)\eta_0^R(x)|.$$

Since

$$|\nabla \eta_0^R(x)| \leq \frac{c}{R}, \quad |\Delta \eta_0^R(x)| \leq \frac{c}{R^2}, \quad |\nabla \mathbb{G}[\chi_{B_r(x_0)}]| \leq \frac{c}{R^{1-N}} \quad \text{and} \quad |\mathbb{G}[\chi_{B_r(x_0)}]| \leq \frac{c}{R^{2-N}},$$

there exists $c_{19} > 0$ such that

$$|(-\Delta)\xi_R(x)| \leq c_{19}R^{-N}$$

for $x \in B_{2R}(x_0) \setminus B_R(x_0)$. Since $u_{k,V}$ is a weak solution, we have

$$\lim_{R \rightarrow +\infty} \sup_{x \in \mathbb{R}^N \setminus B_R(0)} u_{k,V}(x) = 0,$$

which yields

$$\lim_{R \rightarrow +\infty} \int_{B_{2R}(x_0) \setminus B_R(x_0)} u_{k,V}(-\Delta)\xi_R dx = 0.$$

Let $R \rightarrow +\infty$ in (2.12), we see that

$$\int_{B_r(x_0)} u_{k,V} dx = \int_{\mathbb{R}^N} V u_{k,V}^p \mathbb{G}[\chi_{B_r(x_0)}] dx + k\mathbb{G}[\chi_{B_r(x_0)}](0).$$

By (2.11) and the fact that

$$u_{k,V} \geq k\mathbb{G}[\delta_0] \quad \text{and} \quad \mathbb{G}[\chi_{B_r(x_0)}] > 0,$$

we obtain

$$\begin{aligned} \int_{B_r(x_0)} u_{k,V} dx &\geq k^{p-1} \int_{\mathbb{R}^N} V u_{k,V} \mathbb{G}[\delta_0]^{p-1} \mathbb{G}[\chi_{B_r(x_0)}] dx + k\mathbb{G}[\chi_{B_r(x_0)}](0) \\ &\geq c_{20}k^{p-1} \int_{B_r(x_0)} u_{k,V} dx + k\mathbb{G}[\chi_{B_r(x_0)}](0) \end{aligned}$$

with $c_{20} > 0$, which is impossible if k is sufficient large. The assertion follows.

Next, we prove (ii). Let $V_1 \geq V_2$ and u_{k,V_1} be a positive solution of problem (P_k) with $V = V_1$. Therefore, u_{k,V_1} is a super-solution of (P_k) with $V = V_2$. It implies that problem (P_k) with $V = V_2$ has a minimal solution $u_{k,V_2} \leq u_{k,V_1}$. This shows that the mapping $V \mapsto k^*$ is decreasing and the mapping $V \mapsto u_{k,V}$ is increasing.

Finally, we show (iii) is valid. In fact, if V is radially symmetric, so is v_n , which is defined in (2.6) since v_0 is radially symmetric. It follows that the limit $u_{k,V}$ of v_n is radially symmetric too. \square

For future reference, we remark that for $p > 1$ and $k \in (0, k_p]$ with $k_p := (c_2 p)^{-\frac{1}{p-1} \frac{p-1}{p}}$, the minimal solution $u_{k,V}$ verifies

$$u_{k,V} \leq w_{t_p} \leq c_{21}k\mathbb{G}[\delta_0] \quad \text{in} \quad \mathbb{R}^N \setminus \{0\} \quad (2.13)$$

for some $c_{21} > 0$ depending only on k_p . Thus, $V u_{k,V}$ is locally bounded in $\mathbb{R}^N \setminus \{0\}$, which allows us to show that $u_{k,V}$ is a classical solution of (1.9).

3. PROPERTIES OF MINIMAL SOLUTIONS

In this section, we establish the regularity and the decaying law for weak solutions, as well as the stability for the minimal solution.

First, we have the regularity results for weak solutions of (P_k) .

Proposition 3.1. *Assume that the function V satisfies (1.5) with $a_\infty > a_0$ and $a_0 \in \mathbb{R}$, and*

$$p \in \left(0, \frac{N}{N-2}\right). \quad (3.1)$$

Then, any positive weak solution u of (P_k) is a classical solution of (1.9).

Proof. Let u be a weak solution of (P_k) . Since $Vu^p \in L^1(\mathbb{R}^N)$, u can be rewritten as

$$u = \mathbb{G}[Vu^p] + k\mathbb{G}[\delta_0].$$

For any $x_0 \in \mathbb{R}^N \setminus \{0\}$, let $r_0 = \frac{1}{4}|x_0|$. Then, we have that for any $i \in \mathbb{N}$,

$$u = \mathbb{G}[\chi_{B_{2^{-i+1}r_0}(x_0)}Vu^p] + \mathbb{G}[\chi_{\mathbb{R}^N \setminus B_{2^{-i+1}r_0}(x_0)}Vu^p] + k\mathbb{G}[\delta_0]$$

and $V \in L_{loc}^\infty(\mathbb{R}^N \setminus \{0\})$.

For $x \in B_{2^{-i}r_0}(x_0)$, we have that

$$\mathbb{G}[\chi_{\mathbb{R}^N \setminus B_{2^{-i+1}r_0}(x_0)}Vu^p](x) = \int_{\mathbb{R}^N \setminus B_{2^{-i+1}r_0}(x_0)} \frac{c_N V(y)u^p(y)}{|x-y|^{N-2}} dy,$$

then

$$\|\mathbb{G}[\chi_{\mathbb{R}^N \setminus B_{2^{-i}r_0}(x_0)}Vu^p]\|_{C^2(B_{2^{-i+1}r_0}(x_0))} \leq C_i \|Vu^p\|_{L^1(B_{2r_0}(x_0))}, \quad (3.2)$$

where $C_i > 0$ depends on i . Obviously, there holds for some $c_i > 0$ depending on i

$$\|\mathbb{G}[\delta_0]\|_{C^2(B_{2^{-i+1}r_0}(x_0))} \leq c_i |x_0|^{2-N}. \quad (3.3)$$

By (iii) of Proposition 5.1, $Vu^p \in L^{q_0}(B_{2r_0}(x_0))$ with $q_0 = \frac{1}{2}(1 + \frac{1}{p}\frac{N}{N-2}) > 1$. We iterate by Proposition 5.1 that

$$\mathbb{G}[\chi_{B_{2r_0}(x_0)}Vu^p] \in L^{p_1}(B_{2r_0}(x_0)) \text{ with } p_1 = \frac{Nq_0}{N-2q_0}.$$

Similarly,

$$Vu^p \in L^{q_1}(B_{r_0}(x_0)) \text{ with } q_1 = \frac{p_1}{p},$$

and

$$\mathbb{G}[\chi_{B_{r_0}(x_0)}Vu^p] \in L^{p_2}(B_{r_0}(x_0)) \text{ with } p_2 = \frac{Nq_1}{N-2q_1}.$$

Let $q_i = \frac{p_i}{p}$. If $N - 2q_i > 0$, we obtain inductively that

$$Vu^p \in L^{q_i}(B_{2^{-i}r_0}(x_0))$$

and

$$\mathbb{G}[\chi_{B_{2^{-i}r_0}(x_0)}Vu^p] \in L^{p_{i+1}}(B_{2^{-i}r_0}(x_0)) \text{ with } p_{i+1} = \frac{Nq_i}{N-2q_i}.$$

We may verify that

$$\frac{q_{i+1}}{q_i} = \frac{1}{p} \frac{N}{N-2q_i} > \frac{1}{p} \frac{N}{N-2q_1} > 1.$$

Therefore,

$$\lim_{i \rightarrow +\infty} q_i = +\infty.$$

So there exists i_0 such that $N - 2q_{i_0} > 0$, but $N - 2q_{i_0+1} < 0$, and we deduce that

$$\mathbb{G}[\chi_{B_{2^{-i_0}r_0}}(x_0)Vu^p] \in L^\infty(B_{2^{-i_0}r_0}(x_0)).$$

As a result,

$$u(x_0) \leq c_{i_0} \|\mathbb{G}[\delta_0]\|_{L^\infty(B_{2r_0})(x_0)} + c_{i_0} \|Vu^p\|_{L^1(B_{2r_0})(x_0)} \rightarrow 0 \quad \text{as } |x_0| \rightarrow +\infty$$

and

$$Vu^p \in L^\infty(B_{2^{-i_0}r_0}(x_0)).$$

On the other hand, by Proposition 5.2,

$$|\nabla \mathbb{G}[\chi_{B_{2^{-i_0}r_0}}(x_0)Vu^p]| \in L^\infty(B_{2^{-i_0}r_0}(x_0)).$$

By elliptic regularity, we know from (3.3) that u is Hölder continuous in $B_{2^{-i_0}r_0}(x_0)$, so is Vu^p . Hence, u is a classical solution of (1.9). \square

Next, we study the singularity of the weak solution of (P_k) at the origin and the decay at infinity.

Lemma 3.1. *Suppose that the function V satisfies (1.5) with a_0 and a_∞ given in (1.18), and $p > 1$ satisfies (1.17) and (3.1). Let u be a weak solution of (P_k) . Then*

$$\sup_{x \in \mathbb{R}^N \setminus \{0\}} u(x)|x|^{N-2} < +\infty. \quad (3.4)$$

Proof. It is known from Proposition 3.1 that u is a classical solution of (1.9). So we focus on the problems of the singularity of the weak solution at the origin and the decay at infinity.

We first consider the singularity at the origin. We claim that

$$\lim_{|x| \rightarrow 0^+} u(x)|x|^{N-2} = c_N k. \quad (3.5)$$

Indeed, since $\mathbb{G}[Vu^p \chi_{\mathbb{R}^N \setminus B_1(0)}] \in C^2(B_{\frac{1}{2}}(0))$ and $k\mathbb{G}[\delta_0](x) = c_N k|x|^{2-N}$, we see from

$$u = \mathbb{G}[Vu^p \chi_{B_1(0)}] + k\mathbb{G}[\delta_0] + \mathbb{G}[Vu^p \chi_{\mathbb{R}^N \setminus B_1(0)}] \quad (3.6)$$

that it is sufficient to estimate $\mathbb{G}[Vu^p \chi_{B_1(0)}]$ in $B_{\frac{1}{2}}(0)$. Let

$$u_1 = \mathbb{G}[Vu^p \chi_{B_1(0)}].$$

We infer from $Vu^p \in L^{s_0}(B_{\frac{1}{2}}(0))$ with $s_0 = \frac{1}{2}[1 + \frac{N}{p(N-2)}] > 1$ and Proposition 5.1 that $u_1 \in L^{s_1 p}(B_{\frac{1}{2}}(0))$ and $u_1^p \in L^{s_1}(B_{\frac{1}{2}}(0))$ with

$$s_1 = \frac{1}{p} \frac{N}{N - 2s_0} s_0.$$

By (3.6),

$$u^p \leq c_{22}(u_1^p + k^p \mathbb{G}^p[\delta_{x_0}] + 1) \quad \text{in } B_1(0), \quad (3.7)$$

where $c_{22} > 0$. By the definition of u_1 and (3.7), we obtain

$$u_1 \leq c_{22}(\mathbb{G}[u_1^p] + k^p \mathbb{G}[V\mathbb{G}^p[\delta_0]] + \mathbb{G}[\chi_{B_1(0)}]), \quad (3.8)$$

where

$$\mathbb{G}[\chi_{B_1(0)}] \in L^\infty(B_{\frac{1}{2}}(0)), \quad k^p \mathbb{G}[V\mathbb{G}^p[\delta_0]](x) \leq c_{23}|x|^{(2-N)p-a_0+2}$$

and

$$(2-N)p - a_0 + 2 > 2 - N.$$

If $s_1 > \frac{1}{2}Np$, by Proposition 5.1, $u_1 \in L^\infty(B_{2^{-1}}(0))$. Hence, we know from (3.8) that

$$u_1(x) \leq c_{24}|x|^{(2-N)p-a_0+2} \quad (3.9)$$

in $B_{2^{-1}}(0)$. Since $(2 - N)p - a_0 + 2 > 2 - N$, we deduce from (3.6) and (3.9) that (3.5) holds.

On the other hand, if $s_1 < \frac{1}{2}Np$, we proceed as above. Let

$$u_2 = \mathbb{G}[\chi_{B_{2^{-1}}(0)} u_1^p].$$

By Proposition 5.1, $u_2 \in L^{s_2 p}(B_{2^{-1}}(0))$, where

$$s_2 = \frac{1}{p} \frac{Ns_1}{Np - 2s_1} > \frac{N}{N - s_0} s_1 > \left(\frac{1}{p} \frac{N}{N - 2s_0} \right)^2 s_0.$$

Inductively, we define

$$s_m = \frac{1}{p} \frac{Ns_{m-1}}{Np - 2s_{m-1}} > \left(\frac{1}{p} \frac{N}{N - 2s_0} \right)^m s_0.$$

So there is $m_0 \in \mathbb{N}$ such that

$$s_{m_0} > \frac{1}{2}Np$$

and

$$u_{m_0} \in L^\infty(B_{2^{-m_0-1}}(0)).$$

Therefore, (3.5) holds.

Next, we establish the decay at infinity, that is,

$$\limsup_{|x| \rightarrow +\infty} u(x) |x|^{N-2} < +\infty. \quad (3.10)$$

It is known from Proposition 3.1 that $u \in L_{loc}^\infty(\mathbb{R}^N \setminus \{0\})$ and

$$\lim_{|x| \rightarrow +\infty} u(x) = 0. \quad (3.11)$$

We divide the proof into three parts in accordance to a_∞ : (a) $a_\infty > N$; (b) $a_\infty \in (2, N]$; (c) $a_\infty \in (0, 2]$.

Case (a) $a_\infty > N$. Let

$$\psi_0(x) = |x|^{2-N} - |x|^{2-a_\infty} \quad \text{for } |x| \geq 2.$$

There exists $c_{25} > 0$ such that

$$-\Delta \psi_0(x) \geq c_{25} |x|^{-a_\infty}.$$

By (3.11) and the assumption on V , there exist constants $A, B \geq 1$ such that

$$V(x)u^p(x) \leq A|x|^{-a_\infty} \quad \text{if } |x| \geq 2 \quad \text{and} \quad u(x) \leq B(2^{2-N} - 2^{2-a_\infty}) \quad \text{if } |x| = 2.$$

By the comparison principle,

$$u(x) \leq AB\psi_0 \leq AB|x|^{2-N} \quad \text{if } |x| \geq 2.$$

Case (b) $a_\infty \in (2, N]$. Let

$$\tau_1 = \begin{cases} 2 - a_\infty & \text{if } a_\infty \in (2, N), \\ \frac{1}{p}(2 - N) & \text{if } a_\infty = N, \end{cases}$$

and denote

$$\psi_1(x) = |x|^{\tau_1}.$$

Hence, there exists $c_{26} > 0$ such that

$$-\Delta \psi_1(x) \geq c_{26} |x|^{-a_\infty}, \quad x \neq 0.$$

We may find constants $A, B \geq 1$ such that

$$V(x)u^p(x) \leq A|x|^{\tau_1-2} \quad \text{if } |x| \geq 1 \quad \text{and} \quad u(x) \leq B \quad \text{if } |x| \geq 1.$$

By the comparison principle again,

$$u(x) \leq AB\psi_1(x) \quad \text{for } |x| \geq 1.$$

Now, we denote

$$\tau_2 = 2 - a_\infty + p\tau_1.$$

If $\tau_2 \in [-N, -2)$, let

$$\psi_2(x) = |x|^{\tau_2}.$$

Repeating the above argument, we obtain

$$u(x) \leq c_{26}\psi_2(x) \quad \text{if } |x| \geq 1. \quad (3.12)$$

Inductively, we define

$$\tau_j = 2 - a_\infty + p\tau_{j-1}.$$

There exists $j_0 \in \mathbb{N}$ such that $\tau_{j_0-1} > -N$ and $\tau_{j_0} < -N$. If $\tau_{j_0-1} > -N$, we proceed as above. If $\tau_{j_0} < -N$, set

$$\psi_{\tau_{j_0}}(x) = |x|^{2-N} - |x|^{2+\tau_{j_0}}.$$

We reduce the problem to the case (a) $a_\infty > N$. Then, (3.10) holds.

Finally, we consider the case (c) $a_\infty \in (0, 2]$.

For $|x| > 2$ fixed, let $r_0 = \frac{1}{2}|x|^{\frac{a_\infty}{N}}$, where $\frac{a_\infty}{N} \in (0, 1)$. Therefore,

$$\begin{aligned} \mathbb{G}[Vu^p](x) &= \int_{B_{r_0}(x)} \frac{c_N}{|x-y|^{N-2}} V(y)u^p(y)dy + \int_{\mathbb{R}^N \setminus B_{r_0}(x)} \frac{c_N}{|x-y|^{N-2}} V(y)u^p(y)dy \\ &\leq c_{27}(|x| - r_0)^{-a_\infty} \|u\|_{L^\infty(B_{r_0}(x))}^p r_0^2 + r_0^{2-N} \|Vu^p\|_{L^1(\mathbb{R}^N)} \\ &\leq c_{28}|x|^{-(1+\frac{2}{N})a_\infty} \end{aligned}$$

and

$$u = \mathbb{G}[Vu^p] + k\mathbb{G}[\delta_0] \quad \text{and} \quad \mathbb{G}[\delta_0](x) = c_N|x|^{2-N}$$

imply that

$$u \leq c_{29}|x|^{2-N} \quad \text{if } \gamma_0 := (1 + \frac{2}{N})a_\infty \geq N - 2,$$

in this case we are done; and

$$u(x) \leq c_{30}|x|^{-\gamma_0} \quad \text{if } \gamma_0 < N - 2.$$

In the case $a_\infty + p\gamma_0 \leq 2$, let $r_1 = \frac{1}{2}|x|^{\frac{a_\infty + p\gamma_0}{N}}$, where $\frac{a_\infty + p\gamma_0}{N} \in (0, 1)$. We have

$$\begin{aligned} \mathbb{G}[Vu^p](x) &= \int_{B_{r_1}(x)} \frac{c_N}{|x-y|^{N-2}} V(y)u^p(y)dy + \int_{\mathbb{R}^N \setminus B_{r_1}(x)} \frac{c_N}{|x-y|^{N-2}} V(y)u^p(y)dy \\ &\leq c_{31}(|x| - r_1)^{-a_\infty - p\gamma_0} r_1^2 + c_N r^{2-N} \|Vu^p\|_{L^1(\mathbb{R}^N)} \\ &\leq c_{32}|x|^{-(1+\frac{2}{N})(a_\infty + p\gamma_0)}, \end{aligned}$$

which implies that

$$u(x) \leq c_{33}|x|^{-\gamma_1},$$

where $\gamma_1 = (1 + \frac{2}{N})(a_\infty + p\gamma_0)$.

Inductively, we define $r_j = |x|^{\gamma_j}$ with $\gamma_j = (1 + \frac{2}{N})(a_\infty + p\gamma_j)$. There exists $j_0 \in \mathbb{N}$ such that $a_\infty + p\gamma_{j_0-1} \leq 2$ and $a_\infty + p\gamma_{j_0} > 2$. In the former case, we iterate as above; in the latter case, we have

$$V(x)u^p(x) \leq c_{34}|x|^{a_\infty + p\gamma_{j_0}}.$$

By the proof of (b), (3.10) holds. The proof is complete. \square

Now, we deal with the stability of the minimal solution of (P_k) .

Proposition 3.2. *Assume that the function V satisfies (1.5) with $a_0 < \min\{2, a_\infty\}$, $p > 1$ satisfies (1.17) and $k \in (0, k^*)$. Then, any minimal positive solution $u_{k,V}$ of (P_k) is stable. Moreover,*

$$\int_{\mathbb{R}^N} |\nabla \xi|^2 dx - p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx \geq c_3 [(k^*)^{\frac{p-1}{p}} - k^{\frac{p-1}{p}}] \int_{\mathbb{R}^N} |\nabla \xi|^2 dx, \quad \forall \xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}. \quad (3.13)$$

Proof. We divide the proof into two parts. At first, we show the stability of the minimal solution for $k > 0$ small; then, we prove this is true for full range $k \in (0, k^*)$.

Now we show for $k > 0$ small, the result holds true. By (2.13), for $k > 0$ small,

$$u_{k,V}(x) \leq c_{35} k |x|^{2-N} \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where $c_{35} > 0$ is independent of k . Therefore,

$$V(x) u_{k,V}^{p-1}(x) \leq c_{35}^{p-1} c_1 k^{p-1} \frac{|x|^{(2-N)(p-1)-a_0}}{1 + |x|^{a_\infty - a_0}}. \quad (3.14)$$

we note for

$$p \in \left(\frac{N - a_\infty}{N - 2}, \frac{N - a_0}{N - 2} \right),$$

there holds

$$(2 - N)(p - 1) - a_0 \geq -2 \quad \text{and} \quad (2 - N)(p - 1) - a_\infty < -N.$$

It implies from (3.14) that

$$V(x) u_{k,V}^{p-1}(x) \leq c_{35}^{p-1} c_1 \frac{k^{p-1}}{|x|^2}. \quad (3.15)$$

Hence, for any $\xi \in C_c^{1,1}(\mathbb{R}^N)$, by (3.15) and the Hardy-Sobolev inequality, we deduce for $k > 0$ small that

$$\int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx \leq c_{35}^{p-1} c_1 k^{p-1} \int_{\mathbb{R}^N} \frac{\xi^2(x)}{|x|^2} dx \leq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla \xi|^2 dx. \quad (3.16)$$

Inequality (3.16) holds also for $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, which means that $u_{k,V}$ is a semi-stable solution of (P_k) for $k > 0$ small. Indeed, by a density argument, taking $\{\xi_n\}_n$ in $C_c^{1,1}(\mathbb{R}^N)$ so that $\xi_n \rightarrow \xi$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, and replacing ξ in (3.16) by ξ_n , the result follows by passing the limit.

Next, we prove the stability of minimal solutions for all $k \in (0, k^*)$. Suppose that if u_k is not stable, then we have that

$$\lambda_1 := \inf_{\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla \xi|^2 dx}{p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx} \leq 1. \quad (3.17)$$

By the compact embedding theorem in section 4, λ_1 is achieved by a nonnegative function ξ_1 satisfying

$$-\Delta \xi_1 = \lambda_1 p V u_{k,V}^{p-1} \xi_1.$$

Choosing $\hat{k} \in (k, k^*)$ and letting $w = u_{\hat{k},V} - u_{k,V} > 0$, we have that

$$w = \mathbb{G}[V u_{\hat{k},V}^p - V u_{k,V}^p] + (\hat{k} - k) \mathbb{G}[\delta_0].$$

By the elementary inequality

$$(a + b)^p \geq a^p + p a^{p-1} b \quad \text{for } a, b \geq 0,$$

we infer that

$$w \geq \mathbb{G}[V u_{k,V}^{p-1} w] + (\hat{k} - k) \mathbb{G}[\delta_0].$$

Choosing $t > 0$ small, we obtain

$$\begin{aligned} \lambda_1 \int_{\mathbb{R}^N} pV u_{k,V}^{p-1} w \xi_1 dx &= \int_{\mathbb{R}^N} (-\Delta) w \xi_1 dx \\ &\geq \int_{\mathbb{R}^N} pV u_{k,V}^{p-1} w \xi_1 dx + (\hat{k} - k) \xi_1(0) > \int_{\mathbb{R}^N} pV u_{k,V}^{p-1} w \xi_1 dx, \end{aligned}$$

which is impossible. Consequently,

$$p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx < \int_{\mathbb{R}^N} |\nabla \xi|^2 dx, \quad \forall \xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}.$$

Finally, we prove (3.13). For any $k \in (0, k^*)$, let $k' = \frac{k+k^*}{2} > k$ and $l_0 = (\frac{k}{k'})^{\frac{1}{p}} < 1$. Hence, there exists a minimal solution $u_{k',V}$ of (P_k) for $k' < k^*$, and the minimal solution $u_{k',V}$ is stable. Noting that $k - k'l_0^p = 0$, we deduce

$$\begin{aligned} l_0 u_{k',V} &\geq l_0^p u_{k',V} \\ &= l_0^p \left(\mathbb{G}[V u_{k',V}^p] + k' \mathbb{G}[\delta_0] \right) + (k - k'l_0^p) \mathbb{G}[\delta_0] \\ &= \mathbb{G}[V (l_0 u_{k',V})^p] + k \mathbb{G}[\delta_0], \end{aligned}$$

that is, $l_0 u_{k',V}$ is a super-solution of (P_k) for k . Therefore,

$$l_0 u_{k',V} \geq u_{k,V}.$$

So for $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$,

$$\begin{aligned} 0 &< \int_{\mathbb{R}^N} |\nabla \xi|^2 dx - p \int_{\mathbb{R}^N} V u_{k',V}^{p-1} \xi^2 dx \\ &\leq \int_{\mathbb{R}^N} |\nabla \xi|^2 dx - p l_0^{1-p} \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx \\ &= l_0^{1-p} \left[l_0^{p-1} \int_{\mathbb{R}^N} |\nabla \xi|^2 dx - p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx \right]. \end{aligned}$$

It implies

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla \xi|^2 dx - p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx \\ &= (1 - l_0^{p-1}) \int_{\mathbb{R}^N} |\nabla \xi|^2 dx + \left[l_0^{p-1} \int_{\mathbb{R}^N} |\nabla \xi|^2 dx - p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} \xi^2 dx \right] \\ &\geq (1 - l_0^{p-1}) \int_{\mathbb{R}^N} |\nabla \xi|^2 dx, \end{aligned}$$

which together with the fact

$$1 - l_0^{p-1} \geq c_{36} [(k^*)^{\frac{p-1}{p}} - k^{\frac{p-1}{p}}],$$

implies (3.13). The proof is complete. \square

Corollary 3.1. *Assume that $p > 1$, the function V satisfies (1.5) with $a_0 < \min\{2, a_\infty\}$. Then, for $k \in (0, k_p)$, the minimal solution $u_{k,V}$ of (P_k) is stable and satisfies (1.9) as well as (3.13).*

Proof. Since for $k \leq k_p$, the minimal solution $u_{k,V}$ of (P_k) is controlled by w_{t_p} , this yields $V u_{k,V} \in L_{loc}^\infty(\mathbb{R}^N \setminus \{0\})$. It follows by Proposition 5.1 and Proposition 5.2 that $u_{k,V}$ is a classical solution of (1.9). The rest results follow by the proof of Proposition 3.2 and (2.13). \square

Proof of Theorem 1.2. The assertions in the theorem follow by Proposition 3.1, Proposition 3.2 and Corollary 3.1. \square

4. MOUNTAIN-PASS SOLUTION

In order to find the second solution of (P_k) , we try to find a nontrivial function u so that $u_{k,V} + u$ is a solution of (P_k) , which is different from the minimal solution $u_{k,V}$ of (P_k) . We are then led to consider the problem

$$\begin{aligned} -\Delta u &= V(u_{k,V} + u_+)^p - Vu_{k,V}^p \quad \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow +\infty} u(x) &= 0. \end{aligned} \quad (4.1)$$

It is adequate to find a nontrivial solution of (4.1). Intuitively, the cancelation of the singularity of $u_{k,V}$ in the nonlinear term on the right hand side of (4.1) allows us to find a solution of (4.1) as a critical point of the functional

$$E(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} VF(u_{k,V}, v_+) dx \quad (4.2)$$

defined on $\mathcal{D}^{1,2}(\mathbb{R}^N)$, where

$$F(s, t) = \frac{1}{p+1} [(s + t_+)^{p+1} - s^{p+1} - (p+1)s^p t_+]. \quad (4.3)$$

Let V_0 be given in (1.5) and denote by $L^q(\mathbb{R}^N, V_0 dx)$ the weighted L^q space defined by

$$L^q(\mathbb{R}^N, V_0 dx) = \{u : \int_{\mathbb{R}^N} V_0 |u|^q dx < +\infty\}.$$

We may verify by the following lemma that the functional E is well-defined on $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Lemma 4.1. *Let $a_0 < 2$, $a_\infty > \max\{a_0, 0\}$ and $p > 1$ satisfy (1.17). Then the inclusion $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N, V_0 dx)$ is continuous and compact.*

Proof. For $\beta \in (0, 2)$, it follows by Hölder's inequality, the Hardy inequality and the Sobolev inequality that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{\xi^{2^*(\beta)}}{|x|^\beta} dx &\leq \left(\int_{\mathbb{R}^N} \frac{\xi^2}{|x|^2} dx \right)^{\frac{\beta}{2}} \left(\int_{\mathbb{R}^N} \xi^{2^*} dx \right)^{\frac{2-\beta}{2}} \\ &\leq c_{36} \left(\int_{\mathbb{R}^N} |\nabla \xi|^2 dx \right)^{\frac{\beta}{2} + \frac{2-\beta}{2} \frac{2^*}{2}} = c_{36} \left(\int_{\mathbb{R}^N} |\nabla \xi|^2 dx \right)^{\frac{2^*(\beta)}{2}}. \end{aligned} \quad (4.4)$$

We claim that the inclusion $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, V_0 dx)$ is continuous if

$$\max\{2^*(a_\infty), 1\} \leq q \leq \min\{2^*(a_0), 2^*\}.$$

For $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, if $0 \leq a_0 < 2$, by (4.4),

$$\|\xi\|_{L^{2^*(a_0)}(B_1(0), |x|^{a_0} dx)} \leq \|\xi\|_{L^{2^*(a_0)}(\mathbb{R}^N, |x|^{a_0} dx)} \leq c_{36} \|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.$$

If $a_0 < 0$, by Sobolev inequality,

$$\|\xi\|_{L^{2^*}(B_1(0), |x|^{a_0} dx)} \leq \|\xi\|_{L^{2^*}(B_1(0))} \leq c_{36} \|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.$$

Using Hölder's inequality, we obtain

$$\|\xi\|_{L^q(B_1(0), |x|^{a_0} dx)} \leq c_{36} \|\xi\|_{L^{2^*(a_0)}(B_1(0), |x|^{a_0} dx)} \leq c_{36} \|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}.$$

Moreover, for $a_\infty \in (0, 2]$, we have $2^*(a_\infty) < q \leq 2^*$, by Hölder's inequality and (4.4),

$$\int_{\mathbb{R}^N \setminus B_1(0)} |\xi|^q |x|^{-a_\infty} dx \leq \int_{\mathbb{R}^N \setminus B_1(0)} |\xi|^q |x|^{-\tau} dx \leq \|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^{\frac{q}{2}},$$

where $\tau = N - \frac{q(N-2)}{2} < a_\infty$. The case $a_\infty > 2$ can be reduced to the case $a_\infty \in (0, 2]$.

In conclusion, for all the cases, we have

$$\|\xi\|_{L^q(B_1(0), |x|^{a_0} dx)} \leq c_{36} \|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \quad \|\xi\|_{L^{2^*}(\mathbb{R}^N \setminus B_1(0), |x|^{a_\infty} dx)} \leq \|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}. \quad (4.5)$$

Combining with the fact that

$$\lim_{t \rightarrow 0^+} V_0(t) t^{a_0} = c_1 \quad \text{and} \quad \lim_{t \rightarrow +\infty} V_0(t) t^{a_\infty} = c_1,$$

then the claim is true.

We next show that the inclusion $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, V_0 dx)$ is compact if

$$\max\{2^*(a_\infty), 1\} < q < \min\{2^*(a_0), 2^*\}. \quad (4.6)$$

Let $\{\xi_n\}_n$ be a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. For any $\varepsilon > 0$, there exists $R > 0$ such that for $\tau = N - \frac{q(N-2)}{2} < a_\infty$,

$$\int_{\mathbb{R}^N \setminus B_R(0)} |\xi_n|^q |x|^{-a_\infty} dx \leq R^{-a_\infty + \tau} \int_{\mathbb{R}^N \setminus B_R(0)} |\xi_n|^q |x|^{-\tau} dx \leq c R^{-a_\infty + \tau} \leq \frac{\varepsilon}{2}. \quad (4.7)$$

By the Sobolev embedding, $\xi_n \rightarrow \xi$ in $H^1(B_R(0))$ up to a subsequence. This, together with (4.7), yields the result. \square

Corollary 4.1. *The inclusion $id: \mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, V_0 u_{k,V}^{p-1} dx)$ is continuous and compact if $k \leq k_p$.*

Proof. Since $u_{k,V}(x) \leq c_{21} c_N k |x|^{2-N}$ if $k \leq k_p$, there exists $c_{38} > 0$ such that

$$\limsup_{|x| \rightarrow 0^+} u_{k,V}^{p-1}(x) V_0(x) |x|^{a_0 + (p-1)(N-2)} \leq c_{38}$$

and

$$\limsup_{|x| \rightarrow +\infty} u_{k,V}^{p-1}(x) V_0(x) |x|^{a_\infty + (p-1)(N-2)} \leq c_{38},$$

By the proof of Lemma 4.1, we see that the inclusion id is continuous and compact if

$$\max\{2^*(a_\infty + (p-1)(N-2)), 1\} < q < \min\{2^*(a_0 + (p-1)(N-2)), 2^*\}. \quad (4.8)$$

This is the case if $q = 2$. The assertion follows. \square

Proof of Theorem 1.3. Now we prove Theorem 1.3 by the Mountain Pass Theorem.

For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$0 \leq F(s, t) \leq (p + \epsilon) s^{p-1} t^2 + C_\epsilon t^{p+1}, \quad s, t \geq 0$$

By (1.5) and Lemma 4.1, for any $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$,

$$\begin{aligned} \int_{\mathbb{R}^N} V F(u_{k,V}, v_+) dx &\leq (p + \epsilon) \int_{\mathbb{R}^N} V u_{k,V}^{p-1} v_+^2 dx + C_\epsilon \int_{\mathbb{R}^N} V v_+^{p+1} dx \\ &\leq c_\epsilon \|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \end{aligned} \quad (4.9)$$

where $c_\epsilon > 0$. It implies E is well-defined and we verify that E is C^1 on $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Let $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be such that $\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = 1$. For $k \in (0, k_p)$ and $\epsilon > 0$ small enough, we infer from Corollary 3.1 that

$$\begin{aligned} E(tv) &= \frac{1}{2} \|tv\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} VF(u_{k,V}, tv_+) dx \\ &\geq t^2 \left(\frac{1}{2} \|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 - (p+\epsilon) \int_{\mathbb{R}^N} V_0 v_k^{p-1} v^2 dx \right) - C_\epsilon t^{p+1} \int_{\mathbb{R}^N} V_0 |v|^{p+1} dx \\ &\geq c_{39} t^2 \|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 - c_{40} t^{p+1} \|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^{p+1} \\ &= c_{39} t^2 - c_{40} t^{p+1}, \end{aligned}$$

where $c_{39}, c_{40} > 0$. So there exists $t_0 > 0$ small such that

$$E(t_0 v) \geq \frac{c_{39}}{4} t_0^2 =: \beta > 0.$$

On the other hand, we fix a nonnegative function $v_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ with $\|v_0\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = 1$ and its support is a subset of the $\text{supp} V$. Since $(a+b)^p \geq a^p + b^p$ for $a, b > 0$ and $p > 1$,

$$F(u_{k,V}, tv_0) \geq \frac{1}{p+1} \left(t^{p+1} v_0^{p+1} - (p+1) u_{k,V}^p tv_0 \right),$$

There exists $T > 0$ such that for $t \geq T$,

$$\begin{aligned} E(tv_0) &= \frac{t^2}{2} \|v_0\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} VF(u_{k,V}, tv_0) dx \\ &\leq \frac{t^2}{2} \|v_0\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 - \frac{1}{p+1} t^{p+1} \int_{\mathbb{R}^N} V v_0^{p+1} dx + t \int_{\mathbb{R}^N} V u_{k,V}^p v_0 dx \\ &\leq 0. \end{aligned}$$

Choosing $e = Tv_0$, we have $E(e) \leq 0$.

Next, we verify that E satisfies $(PS)_c$ condition, that is, for any sequence $\{v_n\}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $E(v_n) \rightarrow c$ and $E'(v_n) \rightarrow 0$ as $n \rightarrow \infty$, $\{v_n\}$, which is called a $(PS)_c$ sequence, contains a convergent subsequence.

Let $\{v_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$ at the mountain pass level

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} E(\gamma(s)), \quad (4.10)$$

where $\Gamma = \{\gamma \in C([0,1]; \mathcal{D}^{1,2}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = e\}$, and $c \geq \beta$.

By (3.13) and the inequality, see [22, C.2 (iv)],

$$f(s, t)t - (2 + c_p)F(s, t) \geq -\frac{c_p p}{2} s^{p-1} t^2, \quad s, t \geq 0,$$

where $c_p = \min\{1, p-1\}$, we deduce from $E(v_n) \rightarrow c$, $E'(v_n) \rightarrow 0$ that for $c_{41}, c_{42} > 0$,

$$\begin{aligned} c_{41} + c_{41} \|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} &\geq \frac{c_p}{2} \|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} V [(2 + c_p)F(u_{k,V}, (v_n)_+) - f(u_{k,V}, (v_n)_+)(v_n)_+] dx \\ &\geq \frac{c_p}{2} \left[\|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 - p \int_{\mathbb{R}^N} V u_{k,V}^{p-1} v_n^2 dx \right] \\ &\geq c_{42} \frac{c_p}{2} \|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2. \end{aligned}$$

Therefore, v_n is uniformly bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ for $k \in (0, k^*)$. We may assume that there exists $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$v_n \rightharpoonup v \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N), \quad v_n \rightarrow v \quad \text{a.e. in } \mathbb{R}^N.$$

By Lemma 4.1,

$$v_n \rightarrow v \quad \text{in } L^2(\mathbb{R}^N, V_0 u_{k,V}^{p-1} dx) \quad \text{and} \quad L^{p+1}(\mathbb{R}^N, V_0 dx) \quad \text{as } n \rightarrow \infty.$$

Invoking the inequality

$$\begin{aligned} & |F(u_{k,V}, v_n) - F(u_{k,V}, v)| \\ &= \frac{1}{p+1} |(u_{k,V} + (v_n)_+)^p - (u_{k,V} + v_+)^p - (p+1)u_{k,V}^p((v_n)_+ - v_+)| \\ &\leq (p+\epsilon)u_{k,V}^{p-1}((v_n)_+ - v_+)^2 + C_\epsilon((v_n)_+ - v_+)^{p+1}, \end{aligned}$$

we have

$$F(u_{k,V}, v_n) \rightarrow F(u_{k,V}, v) \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad L^1(\mathbb{R}^N, V_0 dx).$$

This, together with $\lim_{n \rightarrow \infty} E(v_n) = c$, implies $\|v_n\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \rightarrow \|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$ as $n \rightarrow \infty$. Hence, $v_n \rightarrow v$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $n \rightarrow \infty$.

By the Mountain Pass Theorem in [2], there exists a nontrivial critical point $v_k \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ of E , which is nonnegative. Thus, v_k is a weak solution of (4.1). Hence,

$$\int_{\mathbb{R}^N} \nabla(u_{k,V} + v_k) \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^N} V(u_{k,V} + v_k)^p \varphi \, dx \quad (4.11)$$

for $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ with $0 \notin \text{supp } \varphi$. We may show as (2.3) in [11] that for any $x_0 \neq 0$ and $r < \frac{1}{2}|x_0|$, there holds

$$\sup_{|x-x_0|<r} |u(x)| = \lim_{q \rightarrow +\infty} \left(\int_{B_r(x_0)} V(x) |u(x)|^q \, dx \right)^{\frac{1}{q}}.$$

By the assumption that $p < \frac{N-a_0}{N-2}$ and $u_{k,V}(x) \leq c_{21}k|x|^{2-N}$, there exists $q > \frac{N}{2}$ such that

$$Vu_{k,V}^{p-1} \in L_{loc}^q(\mathbb{R}^N \setminus \{0\}).$$

Therefore, we deduce by the Moser-Nash iteration, see for instance [11, 14], that $u_{k,V} + v_k \in L_{loc}^\infty(\mathbb{R}^N \setminus \{0\})$, from this $u_{k,V} + v_k \in C_{loc}^2(\mathbb{R}^N \setminus \{0\})$. Moreover, by Theorem 2 in [11] we have

$$\limsup_{|x| \rightarrow +\infty} (u_{k,V} + v_k)|x|^{N-2} < +\infty,$$

implying

$$V(u_{k,V} + v_k)^p \in L^1(\mathbb{R}^N).$$

In conclusion,

$$\int_{\mathbb{R}^N} (u_{k,V} + v_k)(-\Delta)\xi \, dx = \int_{\mathbb{R}^N} V(u_{k,V} + v_k)^p \xi \, dx + k\xi(0), \quad \forall \xi \in C_c^{1,1}(\mathbb{R}^N). \quad (4.12)$$

This means that $v_k + u_{k,V}$ is weak solution of (P_k) , and also implies $v_k + u_{k,V}$ is a classical solution of (1.9). The maximum principle yields $v_k > 0$, and then $v_k + u_{k,V} > u_{k,V}$. So we obtain two positive solutions of (P_k) . \square

Finally, we consider the case that V is radially symmetric. Denote by $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$ the closure of all the radially symmetric functions in $C_c^\infty(\mathbb{R}^N)$ under the norm

$$\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla v|^2 \, dx \right)^{\frac{1}{2}}.$$

Suppose $a_0 < 2$, $a_\infty > \max\{a_0, 0\}$ and p satisfy (1.19), we may show that the inclusion $\mathcal{D}_r^{1,2}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N, V dx)$ is continuous and compact.

Proof of Theorem 1.4. Since V is radially symmetric, so is the minimal solution $u_{k,V}$ of (P_k) for $k \in (0, k_p]$. The solution $u_{k,V}$ is also stable. By the Mountain Pass Theorem, we may find a critical point of the functional

$$E_r(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} VF(u_{k,V}, v_+) dx \quad (4.13)$$

in $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$. The rest of the proof is similar to the proof of Theorem 1.3, we sketch it. \square

5. APPENDIX: REGULARITIES

We recall $G(x, y) = \frac{c_N}{|x-y|^{N-2}}$ the Green kernel of $-\Delta$ in $\mathbb{R}^N \times \mathbb{R}^N$, $\mathbb{G}[\cdot]$ the Green operator defined as

$$\mathbb{G}[f](x) = \int_{\mathbb{R}^N} G(x, y) f(y) dy.$$

Proposition 5.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain and $h \in L^s(\Omega)$. Then, there exists $c_{43} > 0$ such that*

$$(i) \quad \|\mathbb{G}[h]\|_{L^\infty(\Omega)} \leq c_{43} \|h\|_{L^s(\Omega)} \quad \text{if} \quad \frac{1}{s} < \frac{2}{N}; \quad (5.1)$$

$$(ii) \quad \|\mathbb{G}[h]\|_{L^r(\Omega)} \leq c_{43} \|h\|_{L^s(\Omega)} \quad \text{if} \quad \frac{1}{s} \leq \frac{1}{r} + \frac{2}{N} \quad \text{and} \quad s > 1; \quad (5.2)$$

$$(iii) \quad \|\mathbb{G}[h]\|_{L^r(\Omega)} \leq c_{43} \|h\|_{L^1(\Omega)} \quad \text{if} \quad 1 < \frac{1}{r} + \frac{2}{N}. \quad (5.3)$$

Proof. First, we prove (5.1). By Hölder's inequality, for any $x \in \Omega$,

$$\begin{aligned} & \left\| \int_{\Omega} G(x, y) h(y) dy \right\|_{L^\infty(\Omega)} \\ & \leq \left\| \left(\int_{\Omega} G(x, y)^{s'} dy \right)^{\frac{1}{s'}} \left(\int_{\Omega} |h(y)|^s dy \right)^{\frac{1}{s}} \right\|_{L^\infty(\Omega)} \\ & \leq c_N \|h\|_{L^s(\Omega)} \left\| \int_{\Omega} \frac{1}{|x-y|^{(N-2)s'}} dy \right\|_{L^\infty(\Omega)}, \end{aligned}$$

where $s' = \frac{s}{s-1}$. Since $\frac{1}{s} < \frac{2}{N}$, $(N-2)s' < N$, we have

$$\int_{\Omega} \frac{1}{|x-y|^{(N-2)s'}} dy \leq \int_{B_d(x)} \frac{1}{|x-y|^{(N-2)s'}} dy = c_{44} \int_0^d r^{N-1-(N-2)s'} dr \leq c_{45} d^{N-(N-2)s'},$$

where $c_{44}, c_{45} > 0$ and $d = \sup\{|x-y| : x, y \in \Omega\}$. Then (5.1) holds.

Next, we prove (5.2) for $r \leq s$ and (5.3) for $r = 1$. There holds

$$\begin{aligned} & \left\{ \int_{\Omega} \left[\int_{\Omega} G(x, y) h(y) dy \right]^r dx \right\}^{\frac{1}{r}} \\ & = \left\{ \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} G(x, y) h(y) \chi_{\Omega}(x) \chi_{\Omega}(y) dy \right]^r dx \right\}^{\frac{1}{r}} \\ & \leq c_N \left\{ \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} \frac{h(y) \chi_{\Omega}(x) \chi_{\Omega}(y)}{|x-y|^{N-2}} dy \right]^r dx \right\}^{\frac{1}{r}} \\ & = c_N \left\{ \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} \frac{h(x-y) \chi_{\Omega}(x) \chi_{\Omega}(x-y)}{|y|^{N-2}} dy \right]^r dx \right\}^{\frac{1}{r}}. \end{aligned}$$

By the Minkowski's inequality, we have that

$$\begin{aligned}
& \left\{ \int_{\Omega} \left[\int_{\Omega} G(x, y) h(y) dy \right]^r dx \right\}^{\frac{1}{r}} \\
& \leq c_N \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} \frac{h^r(x-y) \chi_{\Omega}(x) \chi_{\Omega}(x-y)}{|y|^{(N-2)r}} dx \right]^{\frac{1}{r}} dy \\
& \leq c_N \int_{\tilde{\Omega}} \left[\int_{\mathbb{R}^N} h^r(x-y) \chi_{\Omega}(x) \chi_{\Omega}(x-y) dx \right]^{\frac{1}{r}} \frac{1}{|y|^{N-2}} dy \\
& \leq c_N \|h\|_{L^r(\Omega)} \leq c_{46} \|h\|_{L^s(\Omega)},
\end{aligned}$$

where $c_{46} > 0$ $\tilde{\Omega} = \{x - y : x, y \in \Omega\}$ is bounded.

Finally, we prove (5.2) in the case $r > s \geq 1$ and $\frac{1}{s} \leq \frac{1}{r} + \frac{2}{N}$, and (5.3) for $r > 1$, $1 < \frac{1}{r} + \frac{2}{N}$

We claim that if $r > s$ and $\frac{1}{r^*} = \frac{1}{s} - \frac{2}{N}$, the mapping $h \rightarrow \mathbb{G}(h)$ is of weak-type (s, r^*) in the sense that

$$|\{x \in \Omega : |\mathbb{G}[h](x)| > t\}| \leq (A_{s,r^*} \frac{\|h\|_{L^s(\Omega)}}{t})^{r^*}, \quad h \in L^s(\Omega), \quad \text{all } t > 0, \quad (5.4)$$

where constant $A_{s,r^*} > 0$.

Denote for $\nu > 0$ that

$$G_0(x, y) = \begin{cases} G(x, y), & \text{if } |x - y| \leq \nu, \\ 0, & \text{if } |x - y| > \nu \end{cases}$$

and $G_{\infty}(x, y) = G(x, y) - G_0(x, y)$. Then, we have

$$|\{x \in \Omega : |\mathbb{G}[h](x)| > 2t\}| \leq |\{x \in \Omega : |\mathbb{G}_0[h](x)| > t\}| + |\{x \in \Omega : |\mathbb{G}_{\infty}[h](x)| > t\}|,$$

where $\mathbb{G}_0[h]$ and $\mathbb{G}_{\infty}[h]$ are defined similar to $\mathbb{G}[h]$.

By the Minkowski's inequality, we deduce

$$\begin{aligned}
|\{x \in \Omega : |\mathbb{G}_0[h](x)| > t\}| & \leq \frac{\|\mathbb{G}_0(h)\|_{L^s(\Omega)}^s}{t^s} \\
& \leq \frac{\|\int_{\Omega} \chi_{B_{\nu}(x-y)} |x - y|^{2-N} |h(y)| dy\|_{L^s(\Omega)}^s}{t^s} \\
& \leq \frac{[\int_{\Omega} (\int_{\Omega} |h(x-y)|^s dx)^{\frac{1}{s}} |y|^{2-N} \chi_{B_{\nu}}(y) dy]^s}{t^s} \\
& \leq \frac{\|h\|_{L^s(\Omega)}^s \int_{B_{\nu}} |x|^{-N+2} dx}{t^s} = c_{47} \frac{\|h\|_{L^s(\Omega)}^s \nu^2}{t^s}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\mathbb{G}_{\infty}[h]\|_{L^{\infty}(\Omega)} & \leq \left\| \int_{\Omega} \chi_{B_{\nu}^c}(x-y) |x - y|^{2-N} |h(y)| dy \right\|_{L^{\infty}(\Omega)} \\
& \leq \left(\int_{\Omega} |h(y)|^s dy \right)^{\frac{1}{s}} \left\| \left(\int_{\Omega \setminus B_{\nu}(y)} |x - y|^{(2-N)s'} dy \right)^{\frac{1}{s'}} \right\|_{L^{\infty}(\Omega)} \\
& \leq c_{48} \|h\|_{L^s(\Omega)} \nu^{2 - \frac{N}{s}},
\end{aligned}$$

where $s' = \frac{s}{s-1}$ if $s > 1$, and if $s = 1$, $s' = \infty$. Choosing $\nu = \left(\frac{t}{c_{48} \|h\|_{L^s(\Omega)}} \right)^{\frac{1}{2 - \frac{N}{s}}}$, we obtain

$$\|\mathbb{G}_{\infty}[h]\|_{L^{\infty}(\Omega)} \leq t,$$

that means

$$|\{x \in \Omega : |\mathbb{G}_{\infty}[h](x)| > t\}| = 0.$$

With this choice of ν , we have that

$$|\{x \in \Omega : |\mathbb{G}[h]| > 2t\}| \leq c_{49} \frac{\|h\|_{L^s(\Omega)}^s \nu^{2s}}{t^s} \leq c_{50} \left(\frac{\|h\|_{L^s(\Omega)}}{t} \right)^{r^*}.$$

The claim for $r > s$ follows by the Marcinkiewicz interpolation theorem. \square

Proposition 5.2. *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain and $h \in L^s(\Omega)$. Then, there exists $c_{51} > 0$ such that*

(i)

$$\|\nabla \mathbb{G}[h]\|_{L^\infty(\Omega)} \leq c_{51} \|h\|_{L^s(\Omega)} \quad \text{if} \quad \frac{1}{s} < \frac{1}{N}; \quad (5.5)$$

(ii)

$$\|\nabla \mathbb{G}[h]\|_{L^r(\Omega)} \leq c_{51} \|h\|_{L^s(\Omega)} \quad \text{if} \quad \frac{1}{s} \leq \frac{1}{r} + \frac{1}{N} \quad \text{and} \quad s > 1; \quad (5.6)$$

(iii)

$$\|\nabla \mathbb{G}[h]\|_{L^r(\Omega)} \leq c_{51} \|h\|_{L^1(\Omega)} \quad \text{if} \quad 1 < \frac{1}{r} + \frac{1}{N}. \quad (5.7)$$

Proof. Since

$$|\nabla \mathbb{G}[h](x)| = \left| \int_{\Omega} \nabla_x G(x, y) h(y) dy \right| \leq \int_{\Omega} |\nabla_x G(x, y)| |h(y)| dy$$

and

$$|\nabla_x G(x, y)| = c_N(N-2)|x-y|^{1-N},$$

then the conclusions follow as the proof of Proposition 5.1. \square

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